

On bipartite graphs of defect at most 4

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Abstract

We consider the bipartite version of the *degree/diameter problem*, namely, given natural numbers $\Delta \geq 2$ and $D \geq 2$, find the maximum number $N^b(\Delta, D)$ of vertices in a bipartite graph of maximum degree Δ and diameter D . In this context, the Moore bipartite bound $M^b(\Delta, D)$ represents an upper bound for $N^b(\Delta, D)$.

Bipartite graphs of maximum degree Δ , diameter D and order $M^b(\Delta, D)$ – called *Moore bipartite graphs* – have turned out to be very rare. Therefore, it is very interesting to investigate bipartite graphs of maximum degree $\Delta \geq 2$, diameter $D \geq 2$ and order $M^b(\Delta, D) - \epsilon$ with small $\epsilon > 0$; that is, bipartite $(\Delta, D, -\epsilon)$ -graphs. The parameter ϵ is called the *defect*.

This paper considers bipartite graphs of defect at most 4, and presents all the known such graphs. Bipartite graphs of defect 2 have been studied in the past; if $\Delta \geq 3$ and $D \geq 3$, they may only exist for $D = 3$. However, when $\epsilon > 2$ bipartite $(\Delta, D, -\epsilon)$ -graphs represent a wide unexplored area.

The main results of the paper include several necessary conditions for the existence of bipartite $(\Delta, D, -4)$ -graphs; the complete catalogue of bipartite $(3, D, -\epsilon)$ -graphs with $D \geq 2$ and $0 \leq \epsilon \leq 4$; the complete catalogue of bipartite $(\Delta, D, -\epsilon)$ -graphs with $\Delta \geq 2$, $5 \leq D \leq 187$ ($D \neq 6$) and $0 \leq \epsilon \leq 4$; and a proof of the non-existence of all bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 3$ and odd $D \geq 5$.

Finally, we conjecture that there are no bipartite graphs of defect 4 for $\Delta \geq 3$ and $D \geq 5$, and comment on some implications of our results for the upper bounds of $N^b(\Delta, D)$.

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1 Introduction

Due to the diverse features and applications of interconnection networks, it is possible to find many interpretations of network “optimality” in the literature. Here we are concerned with the following; see [8, pp. 18], [10, pp. 168], and [16, pp. 91].

An optimal network contains the maximum possible number of nodes, given a limit on the number of connections attached to a node and a limit on the distance between any two nodes of the network.

This interpretation has attracted network designers and the research community in general due to its implications in the design of large interconnection networks. In graph-theoretical terms, this interpretation leads to the *degree/diameter problem* (the problem of finding the largest possible number of vertices in a graph with given maximum degree and diameter). If the graphs in question are subject to further restrictions such as being bipartite, planarity and/or transitivity, we can state the degree/diameter problem for the classes of graphs under consideration.

In this paper we will consider only bipartite graphs, and in this case, the degree/diameter problem can be stated as follows.

Degree/diameter problem for bipartite graphs: Given natural numbers $\Delta \geq 2$ and $D \geq 2$, find the largest possible number $N^b(\Delta, D)$ of vertices in a bipartite graph of maximum degree Δ and diameter D .

Note that $N^b(\Delta, D)$ is well defined for $\Delta \geq 2$ and $D \geq 2$. An upper bound for $N^b(\Delta, D)$ is given by the *Moore bipartite bound* $M^b(\Delta, D)$, defined below:

$$M^b(\Delta, D) = 2 \left(1 + (\Delta - 1) + \cdots + (\Delta - 1)^{D-1} \right).$$

Bipartite graphs of degree Δ , diameter D and order $M^b(\Delta, D)$ are called *Moore bipartite graphs*. Moore bipartite graphs are rare; for $\Delta = 2$ they are the cycles of length $2D$, while for $\Delta \geq 3$ Moore bipartite graphs exist only for diameters 2, 3, 4 and 6; see [9]. Therefore, we are interested in studying the existence or otherwise of bipartite graphs of given maximum degree Δ , diameter D and order $M^b(\Delta, D) - \epsilon$ for $\epsilon > 0$;

that is, bipartite $(\Delta, D, -\epsilon)$ -graphs, where the parameter ϵ is called the *defect*. For notational convenience, we consider Moore bipartite graphs as having defect $\epsilon = 0$.

Only a few values of $N^b(\Delta, D)$ are known at present. With the exception of $N^b(3, 5) = M^b(3, 5) - 6$, settled in [11], the other known values of $N^b(\Delta, D)$ are those for which there is a Moore bipartite graph. The paper [14] combined with [5, 6] almost settled the case of bipartite graphs of defect 2; if $\Delta \geq 3$ and $D \geq 3$, then such graphs may only exist for $D = 3$ and certain values of Δ . Bipartite $(\Delta, D, -\epsilon)$ -graphs with $\epsilon > 2$ have been rarely considered in the literature so far.

In this paper we consider bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 2$ and $D \geq 3$. By using combinatorial approaches we obtain several important results about bipartite graphs of defect 4, including several necessary conditions for the existence of bipartite $(\Delta, D, -4)$ -graphs; the complete catalogue of bipartite $(3, D, -\epsilon)$ -graphs with $D \geq 2$ and $0 \leq \epsilon \leq 4$; the complete catalogue of bipartite $(\Delta, D, -\epsilon)$ -graphs with $\Delta \geq 2$, $5 \leq D \leq 187$ ($D \neq 6$) and $0 \leq \epsilon \leq 4$; and a proof of the non-existence of all bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 3$ and odd $D \geq 5$. Finally, we conjecture that there are no bipartite graphs of defect 4 for $\Delta \geq 3$ and $D \geq 5$.

The main results in this paper do not apply to bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 4$ and $D = 3, 4$. Some of our assertions, however, do offer a partial characterisation of all bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 3$ and $D \geq 3$. At the time of writing the paper we do not foresee a conclusive way to take on the diameters 3 and 4. To deal with such graphs it would be necessary to either find different ideas or complement some of the ones presented here. Section 6.1 contains further comments on such diameters.

2 Notation and Terminology

The terminology and notation used in this paper is standard and consistent with that used in [7], so only those concepts that can vary from text to text will be defined.

All graphs considered are simple. The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. The *difference* between the graphs Γ and Γ' , denoted by $\Gamma - \Gamma'$, is the graph with vertex set $V(\Gamma) - V(\Gamma')$ and edge set formed by all the edges with both endvertices in $V(\Gamma) - V(\Gamma')$.

The set of neighbours of a vertex x in Γ is denoted by $N(x)$. For an edge $e = \{x, y\}$ we write $e = xy$, or alternatively $x \sim y$. The set of edges in a graph Γ joining a vertex x in $X \subseteq V(\Gamma)$ to a vertex y in $Y \subseteq V(\Gamma)$ is denoted by $E(X, Y)$; for simplicity, we write $E(x, Y)$ rather than $E(\{x\}, Y)$.

A path of length k is called a *k-path*, and cycle of length k is called a *k-cycle*. A path from a vertex x to a vertex y is denoted by $x - y$. Whenever we refer to paths we mean shortest paths. We will use the following notation for subpaths of a path $P = x_0x_1 \dots x_k$: $x_iPx_j = x_i \dots x_j$, where $0 \leq i \leq j \leq k$. The distance between a vertex x and a vertex y is denoted by $d(x, y)$.

The union of three independent paths of length D with common endvertices is denoted by Θ_D . In a graph Γ a vertex of degree at least 3 is called a *branch vertex* of Γ .

3 Known bipartite $(\Delta, D, -\epsilon)$ -graphs with $\Delta \geq 2$, $D \geq 2$ and $0 \leq \epsilon \leq 4$

For $\Delta = 2$ the Moore bipartite graphs are the cycles on $2D$ vertices, while for $D = 2$ and each $\Delta \geq 3$ they are the complete bipartite graphs of degree Δ and order 2Δ . For $D = 3, 4, 6$ Moore bipartite graphs of degree Δ have been constructed only when $\Delta - 1$ is a prime power [1]. Furthermore, Singleton [15] proved that the existence of a Moore bipartite graph of diameter 3 is equivalent to the existence of a projective plane of order $\Delta - 1$. The question of whether Moore bipartite graphs of diameter 3, 4 or 6 exist for other values of Δ remains open, and represents one of the most famous problems in combinatorics. For other values of $D \geq 2$ and $\Delta \geq 3$ there are no Moore bipartite graphs (see [15, 9]).

When $\Delta = 2$ or $D = 2$ bipartite $(\Delta, D, -\epsilon)$ -graphs with $\epsilon \geq 1$ can be obtained by simple observation. For a given $D \geq 2$ there is only one bipartite $(2, D, -\epsilon)$ -graph with $\epsilon \geq 1$: the path of length D , which has defect $\epsilon = D - 1$. For a given $\Delta \geq 2$ there are exactly $\Delta - 1$ bipartite $(\Delta, 2, -\epsilon)$ -graphs with $\epsilon \geq 1$; they are the complete bipartite graphs with partite sets of size Δ and $\Delta - \epsilon$, where $1 \leq \epsilon \leq \Delta - 1$. Therefore, from now on we assume $\Delta \geq 3$ and $D \geq 3$.

We continue with some conditions for the regularity of bipartite $(\Delta, D, -\epsilon)$ -graphs, which were obtained in [5].

Proposition 3.1 ([5]) *For $\epsilon < 1 + (\Delta - 1) + (\Delta - 1)^2 + \dots + (\Delta - 1)^{D-2}$, $\Delta \geq 3$ and $D \geq 3$, a bipartite $(\Delta, D, -\epsilon)$ -graph is regular.*

Proposition 3.2 ([5]) *For $\epsilon < 2((\Delta - 1) + (\Delta - 1)^3 + \dots + (\Delta - 1)^{D-2})$, $\Delta \geq 3$ and odd $D \geq 3$, a bipartite $(\Delta, D, -\epsilon)$ -graph is regular.*

By Propositions 3.1 and 3.2, bipartite $(\Delta, D, -\epsilon)$ -graphs with $\Delta \geq 3$, $D \geq 3$ and $\epsilon \leq 3$ must be regular, implying the non-existence of such graphs for $\Delta \geq 3$, $D \geq 3$ and $\epsilon = 1, 3$. In the same way, bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 3$ and $D \geq 4$ and bipartite $(\Delta, 3, -4)$ -graphs with $\Delta \geq 4$ must be regular.

For $\Delta \geq 3$ and $D \geq 3$, the only known bipartite $(\Delta, D, -2)$ -graphs are depicted in Fig. 1. Recall that such graphs do not exist when $\Delta \geq 3$ and $D \geq 4$; see [5, 6, 14].

Bipartite $(3, 3, -4)$ -graphs may be irregular. Figure 2 depicts all such graphs, which were obtained by using the program *geng* from the package *nauty* written by McKay [13]. The unique bipartite $(3, 4, -4)$ -graph is shown in Fig. 3.

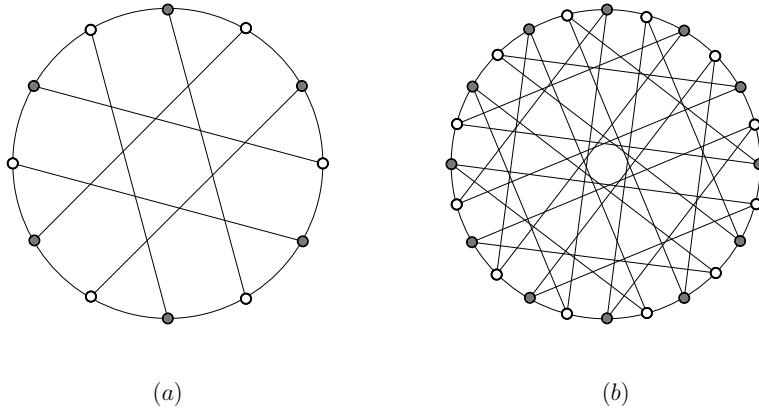


Figure 1: (a) the unique bipartite $(3, 3, -2)$ -graph and (b) the unique bipartite $(4, 3, -2)$ -graph.

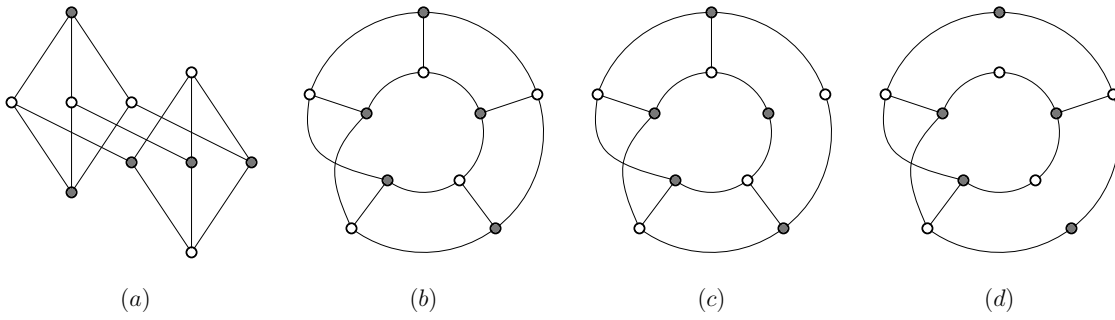


Figure 2: All the bipartite $(3, 3, -4)$ -graphs.

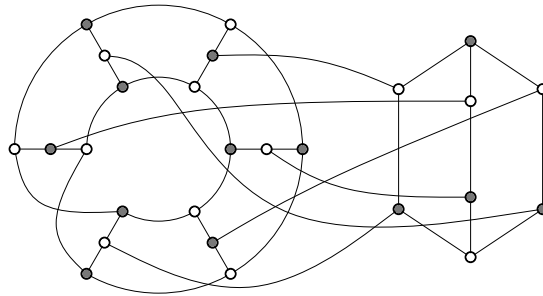


Figure 3: The unique bipartite $(3, 4, -4)$ -graph.

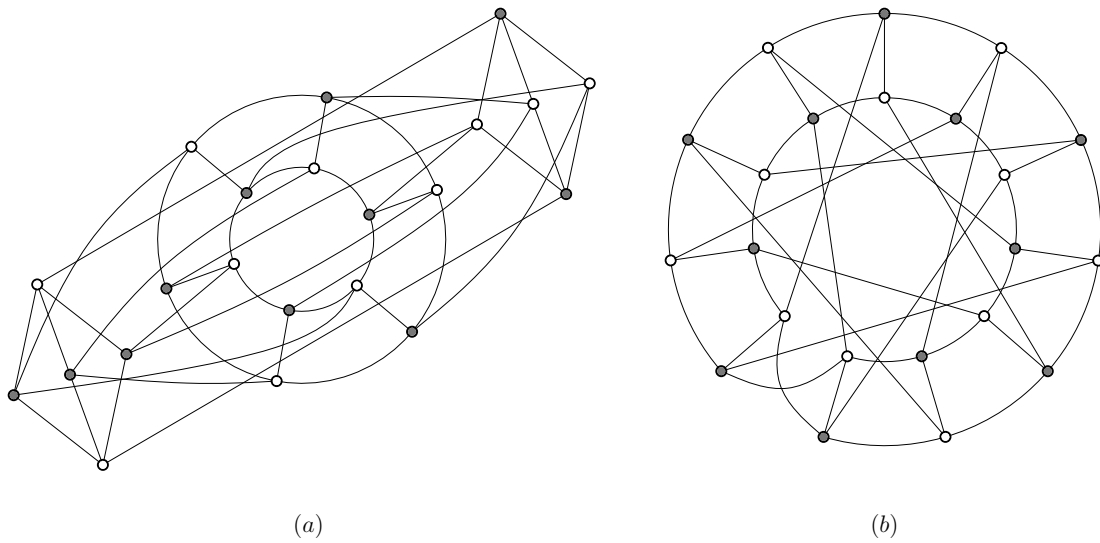


Figure 4: All the bipartite $(4, 3, -4)$ -graphs.

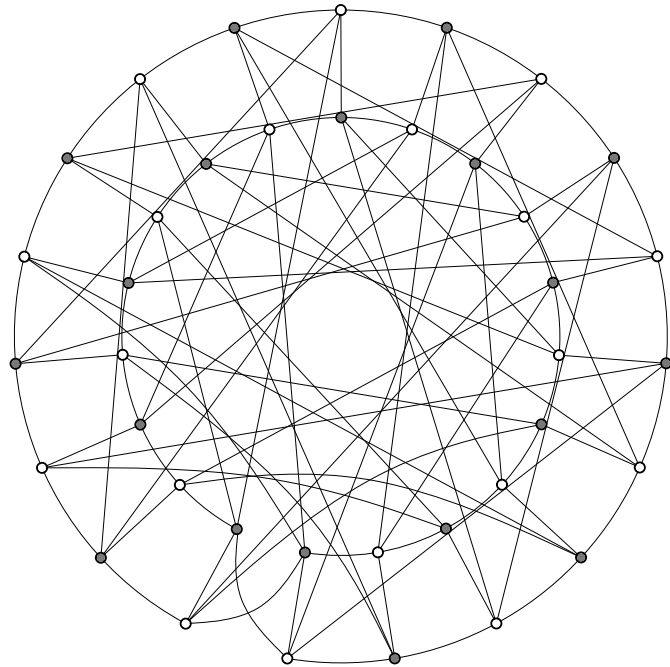


Figure 5: The only known bipartite $(5, 3, -4)$ -graph.

All the bipartite $(4, 3, -4)$ -graphs are depicted in Fig. 4. These graphs were obtained computationally by Meringer [12] using the program *genreg*. An alternative description of the graph in Fig. 4 (b) was communicated to the second author by Charles Delorme: take $\mathbb{Z}/22\mathbb{Z}$ as the vertex set of the graph, and for each even x , add the edges $\{x, x + 1\}$, $\{x, x - 1\}$, $\{x, x + 7\}$ and $\{x, x + 11\}$.

The only known bipartite $(5, 3, -4)$ -graph is depicted in Fig. 5; this graph was independently found by Charles Delorme and by the first author. Charles Delorme described this graph as follows: take $\mathbb{Z}/38\mathbb{Z}$ as its vertex set, and for each even x , add the edges $\{x, x - 1\}$, $\{x, x + 1\}$, $\{x, x + 5\}$, $\{x, x + 13\}$ and $\{x, x + 23\}$.

4 Preliminary Results

From now on, when referring to a class of regular bipartite graphs, we prefer the symbol d to Δ to denote the maximum degree of a graph. However, if the graph class also involves irregular graphs, we use the symbol Δ . Recall that, unless $d = 3$ and $D = 3$, a bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 3$ must be regular. Therefore, when referring to a regular bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 3$, we are actually referring to any bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 3$ other than the ones exhibited in Fig. 2 (c) and (d).

In a bipartite $(d, D, -4)$ -graph we call a cycle of length at most $2D - 2$ a *short cycle*.

Proposition 4.1 *The girth of a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$ is $2D - 2$. Furthermore, any vertex x of Γ lies on the short cycles specified below and no other short cycle, and we have the following cases:*

x is contained in exactly three $(2D - 2)$ -cycles. *Then*

(i) *x is a branch vertex of one Θ_{D-1} , or*

x is contained in two $(2D - 2)$ -cycles. *Then*

(ii) *x lies on exactly two $(2D - 2)$ -cycles, whose intersection is an ℓ -path with $\ell \in \{0, \dots, D - 1\}$.*

Each case is considered as a type. For instance, a vertex satisfying case (i) is called a vertex of Type (i). Note that, if x is of Type (ii) and $\ell = D - 1$, the two short cycles containing x constitute a Θ_{D-1} .

Proof. Let xy be an edge of Γ . Let us use the standard decomposition for a bipartite graph of even girth with respect to the edge xy [3]. For $0 \leq i \leq D - 1$, the sets X_i and Y_i are defined as follows:

$$\begin{aligned}
X_i &= \{z \in V(\Gamma) | d(x, z) = i, d(y, z) = i + 1\} \\
Y_i &= \{z \in V(\Gamma) | d(y, z) = i, d(x, z) = i + 1\}.
\end{aligned}$$

The decomposition of Γ into the sets X_i and Y_i is called the *standard decomposition for a graph of even girth with respect to the edge xy* . Since Γ is bipartite, its girth is even and $X_i \cap Y_j = \emptyset$ for $0 \leq i, j \leq D-1$.

Claim 1 $g(\Gamma) = 2D - 2$.

Proof of Claim 1. Since the assertion is trivial for $D = 3$, we suppose that $g(\Gamma) \leq 2D - 4$ for $D \geq 4$. Assume that the edge xy lies on a cycle of length $g(\Gamma)$. Then, $|X_i| = |Y_i| = (d-1)^i$ for $1 \leq i \leq \frac{g(\Gamma)}{2} - 1$, and

$$\begin{aligned}
|X_{D-2}| &\leq ((d-1)^{D-3} - 1)(d-1) + d - 2 = (d-1)^{D-2} - 1 \\
|Y_{D-2}| &\leq ((d-1)^{D-3} - 1)(d-1) + d - 2 = (d-1)^{D-2} - 1 \\
|X_{D-1}| &\leq ((d-1)^{D-2} - 1)(d-1) \\
|Y_{D-1}| &\leq ((d-1)^{D-2} - 1)(d-1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|V(\Gamma)| &= \sum_{i=0}^{D-1} |X_i| + \sum_{i=0}^{D-1} |Y_i| \leq 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-3}) + \\
&\quad + 2(d-1)^{D-2} - 2 + 2(d-1)^{D-1} - 2(d-1) = \\
&= 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-1}) - 2(d-1) - 2 = \\
&= M^b(d, D) - 2d,
\end{aligned}$$

which is a contradiction. Hence, $g(\Gamma) \geq 2D - 2$. If $g(\Gamma) = 2D$ then the order of Γ would be at least $M^b(d, D)$ [2]. Thus, $g(\Gamma) = 2D - 2$ and the claim follows. \square

We now proceed to prove the second part of the proposition.

For a given vertex x , we use again the standard decomposition for a bipartite graph with respect to an edge xy in Γ . Suppose that there are at least three edges joining vertices at X_{D-2} to vertices at Y_{D-2} ; that is, $|E(X_{D-2}, Y_{D-2})| \geq 3$. In such case

$$\begin{aligned}
|X_{D-1}| &\leq (d-1)^{D-1} - 3, \\
|Y_{D-1}| &\leq (d-1)^{D-1} - 3,
\end{aligned}$$

and therefore

$$\begin{aligned}
|V(\Gamma)| &= \sum_{i=0}^{D-1} |X_i| + \sum_{i=0}^{D-1} |Y_i| \leq 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-2}) + \\
&\quad + 2((d-1)^{D-1} - 3) = \\
&= 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-1}) - 6 = \\
&= M_{d,D}^b - 6,
\end{aligned}$$

which is a contradiction. Consequently, $0 \leq |E(X_{D-2}, Y_{D-2})| \leq 2$.

Suppose that $|E(X_{D-2}, Y_{D-2})| = 2$. If the two edges are both incident to a common vertex of Y_{D-2} then x is of Type (i), otherwise x is of Type (ii).

If instead $|E(X_{D-2}, Y_{D-2})| = 1$ then $|E(X_{D-2}, X_{D-1})| = |E(Y_{D-2}, Y_{D-1})| = (d-1)^{D-1} - 1$. Since $|X_{D-1}| = |Y_{D-1}| = (d-1)^{D-1} - 2$, there is a vertex $u \in X_{D-1}$ such that $|E(u, X_{D-2})| = 2$. Therefore, it follows (ii).

Finally, if $|E(X_{D-2}, Y_{D-2})| = 0$ then both types may occur. Indeed, if there is a vertex $u \in X_{D-1}$ such that $|E(u, X_{D-2})| = 3$ then x is of Type (i) (this case can only occur if $d \geq 4$), otherwise there must exist two vertices $u, v \in X_{D-1}$ such that $|E(u, X_{D-2})| = |E(v, X_{D-2})| = 2$, in which case x is of Type (ii). This completes the proof of the proposition. \square

We continue with the following observation, which will be implicitly used throughout the paper:

Observation 4.1 *Let $\Gamma = (V_1 \cup V_2, E)$ (the sets V_1 and V_2 are called partite sets) be any bipartite graph of even (odd) finite diameter D . The distance between a vertex $u \in V_1$ and any vertex $v \in V_2$ ($w \in V_1$) is at most $D - 1$.*

In virtue of Proposition 4.1, we define the following concepts:

If two short cycles C^1 and C^2 are non-disjoint we say that C^1 and C^2 are *neighbours*.

For a vertex x lying on a short cycle C , we denote by $\text{rep}^C(x)$ the vertex x' in C such that $d(x, x') = D - 1$. We say x' is the *repeat* of x in C and vice versa, or simply that x and x' are *repeats* in C . Alternatively, and more generally, we say that x' is a repeat of x with *multiplicity* $m_x(x')$ ($1 \leq m_x(x') \leq 2$) if there are exactly $m_x(x') + 1$ different paths of length $D - 1$ from x to x' . Proposition 4.1 tells us that a vertex in Γ may have a repeat of multiplicity 2. Accordingly, we denote by $\text{Rep}(x)$ the multiset of the repeats of a vertex x in Γ .

The concept of repeat can be easily extended to paths. For a path $P = x - y$ of length at most $D - 2$ contained in a short cycle C , we denote by $\text{rep}^C(P)$ the path $P' \subset C$ defined as $\text{rep}^C(x) - \text{rep}^C(y)$. We say that P' is the *repeat* of P in C and vice versa, or simply that P and P' are *repeats* in C .

Often our arguments revolve around the identification of the elements in the set S_x of short cycles containing a given vertex x ; we call this process *saturating* the vertex x . A vertex x is called *saturated* if the elements in S_x have been completely identified. The following lemma will help us in this cycle identification process.

Lemma 4.1 (Saturating Lemma) *Let \mathcal{C} be a $(2D - 2)$ -cycle in a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$, and α, α' two vertices in \mathcal{C} such that $\alpha' = \text{rep}^{\mathcal{C}}(\alpha)$. Let γ be a neighbour of α not contained in \mathcal{C} , and $\mu_1, \mu_2, \dots, \mu_{d-2}$ the neighbours of α' not contained in \mathcal{C} . Suppose there is no short cycle in Γ containing the edge $\alpha \sim \gamma$ and intersecting \mathcal{C} at a path of length greater than $D - 3$.*

Then, in Γ there exist a vertex $\mu \in \{\mu_1, \mu_2, \dots, \mu_{d-2}\}$ and a short cycle \mathcal{C}^1 such that γ and μ are repeats in \mathcal{C}^1 , and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$.

Proof. Let α'_1, α'_2 be the neighbours of α' contained in \mathcal{C} .

For $1 \leq i \leq d - 2$, consider the path $P^i = \gamma - \mu_i$. As $g(\Gamma) = 2D - 2$, P^i cannot go through α'_1 or α'_2 . If P^i went through certain μ_j ($j \neq i$), then a cycle $\gamma P^i \mu_j \alpha' \mathcal{C} \alpha \gamma$ would either have length smaller than $2D - 2$ or be a short cycle intersecting \mathcal{C} at a $(D - 1)$ -path. Consequently, P^i must go through one of the neighbours of μ_i other than α' , and must be a $(D - 1)$ -path; see Fig. 6 (a). In addition, $V(P^i \cap \mathcal{C}) = \emptyset$.

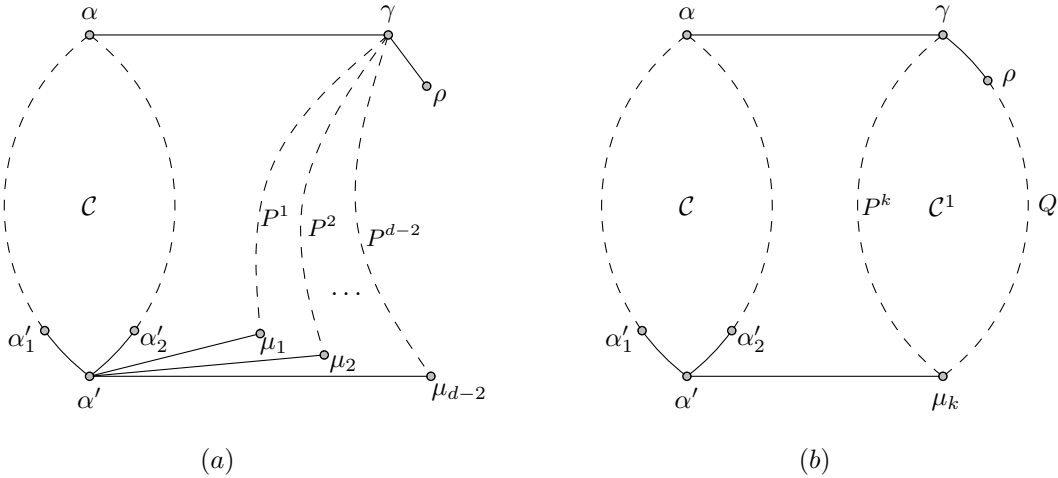


Figure 6: Auxiliary figure for Lemma 4.1

Let ρ be one of the neighbours of γ other than α , not contained in any of the paths P^i (there is at least one of such vertices). Consider a path $Q = \rho - \alpha'$. If Q went through α'_1 , then the closed walk $\rho Q \alpha'_1 \mathcal{C} \alpha \gamma \rho$ would either contain a cycle of length smaller than $2D - 2$ or be a short cycle intersecting \mathcal{C} at a $(D - 2)$ -path. Consequently, Q must go through a certain μ_k ($1 \leq k \leq d - 2$) and $V(Q \cap \mathcal{C}) = \{\alpha'\}$ (Fig. 6 (b)). Note that Q must be a $(D - 1)$ -path, and that $V(Q \cap P^k) = \{\mu_k\}$; otherwise there would be a cycle in Γ of length smaller than $2D - 2$.

Thus, we have obtained a short cycle $\mathcal{C}^1 = \gamma\rho Q\mu_k P^k\gamma$ such that γ and μ_k are repeats in \mathcal{C}^1 , and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$. By setting $\mu = \mu_k$ the lemma follows. \square

Corollary 4.1 *Let α, γ be vertices in a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$ such that $\gamma \in N(\alpha)$. Then, for every $\alpha' \in \text{Rep}(\alpha)$ it follows that $N(\alpha')$ contains a repeat of γ .*

Proof. Let \mathcal{C} be a short cycle containing α and α' . If the vertex γ is contained in \mathcal{C} or the edge $\alpha\gamma$ belongs to a short cycle in Γ intersecting \mathcal{C} at a path of length $D - 2$ or $D - 1$, then the corollary trivially follows. If we instead assume that $\gamma \notin \mathcal{C}$ and there is no short cycle in Γ containing the edge $\alpha\gamma$ and intersecting \mathcal{C} at a path of length greater than $D - 3$, then the corollary follows from the Saturating Lemma. \square

4.1 Repeats of Cycles

In this section we extend the concept of repeat to short cycles; see the Repeat Cycle Lemma.

Lemma 4.2 (Repeat Cycle Lemma) *Let C be a short cycle in a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$, $\{C^1, C^2, \dots, C^k\}$ the set of neighbours of C , and $I_i = C^i \cap C$ for $1 \leq i \leq k$. Suppose at least one I_j , for $j \in \{1, \dots, k\}$, is a path of length smaller than $D - 2$. Then there is an additional short cycle C' in Γ intersecting C^i at $I'_i = \text{rep}^{C^i}(I_i)$, where $1 \leq i \leq k$.*

Proof. Observe that, according to our premises and Proposition 4.1, $k \geq 3$ and $I_i \cap I_j = \emptyset$ for $1 \leq i < j \leq k$. We assume the denotation of the neighbours C^1, C^2, \dots, C^k of C and the corresponding intersection paths $I_1 = x_1 - y_1, I_2 = x_2 - y_2, \dots, I_k = x_k - y_k$ is such that $C = x_1 I_1 y_1 x_2 I_2 y_2 \dots x_k I_k y_k x_1$. For $1 \leq i \leq k$, we also denote the endvertices of I'_i by x'_i and y'_i , where $x'_i = \text{rep}^{C^i}(x_i)$ and $y'_i = \text{rep}^{C^i}(y_i)$ (see Fig. 7 (a)).

For $1 \leq i \leq k$, consider the cycles C^i and $C^{(i \bmod k)+1}$.

First suppose that I_i is a path of length smaller than $D - 2$. Since y_i is saturated, there cannot be a short cycle in Γ , other than C , containing the edge $y_i \sim x_{(i \bmod k)+1}$. Since I_i is a path of length smaller than $D - 2$, we apply the Saturating Lemma (mapping C^i to \mathcal{C} , y_i to α , y'_i to α' and $x_{(i \bmod k)+1}$ to γ) and obtain an additional short cycle \mathcal{C}^1 in Γ such that $x_{(i \bmod k)+1}$ is a repeat in \mathcal{C}^1 of a neighbour $v \notin C^i$ of y'_i , and $\mathcal{C}^1 \cap C_i = \emptyset$. Since $x_{(i \bmod k)+1}$ is saturated, we have that necessarily $\mathcal{C}^1 = C^{(i \bmod k)+1}$, which in turn implies $v = x'_{(i \bmod k)+1}$. In other words, it follows that $y'_i \sim x'_{(i \bmod k)+1} \in E(\Gamma)$.

If instead I_i is a $(D - 2)$ -path then $I_{(i \bmod k)+1}$ must be a path of length smaller than $D - 2$. Therefore, we can apply the above reasoning and deduce that $x'_{(i \bmod k)+1} \sim y'_i \in E(\Gamma)$.

In this way we obtain a subgraph $\Upsilon = \bigcup_{i=1}^k (I'_i \cup y'_i \sim x'_{(i \bmod k)+1}) = x'_1 I'_1 y'_1 x'_2 I'_2 y'_2 \dots x'_k I'_k y'_k x'_1$ intersecting C^i at I'_i for $1 \leq i \leq k$ (see Fig. 7 (b), where part of the subgraph Υ is highlighted in bold).

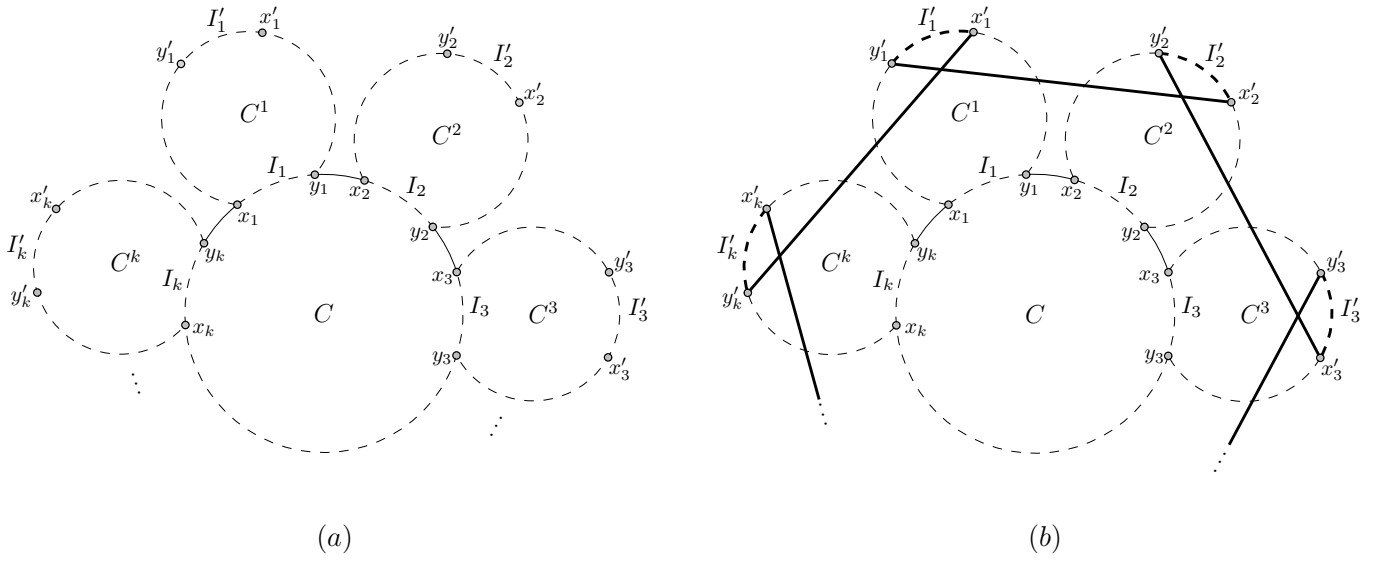


Figure 7: Auxiliary figure for Lemma 4.2

We next show that Υ must indeed be a cycle.

Claim 1. Υ is a $(2D - 2)$ -cycle.

Proof of Claim 1.

If the paths I_i' are pairwise disjoint then Υ is obviously a $(2D - 2)$ -cycle.

Suppose the paths I_i' are not pairwise disjoint; then $|V(\Upsilon)| < 2D - 2$ and, according to Proposition 4.1, Υ contains no cycle. Since Υ is clearly connected it is therefore a tree.

Let $z \in C^\ell$ be an arbitrary leaf in Υ . If the repeat path $I_\ell' = x_\ell' - y_\ell'$ had length greater than 0, then z would have at least two neighbours in Υ . Therefore, $I_\ell = C \cap C^\ell$ contains exactly one vertex, and thus, $x_\ell = y_\ell$ and $z = x_\ell' = y_\ell'$.

Recall we do addition modulo k on the subscripts of the vertices and the superscripts of the cycles.

Since $x_\ell' \sim y_{\ell-1}'$ and $x_\ell' \sim x_{\ell+1}'$ are edges in Υ , it holds that $y_{\ell-1}'$ and $x_{\ell+1}'$ denote the same vertex. Let $u_{\ell-1}', v_{\ell-1}'$ be the neighbours of $y_{\ell-1}'$ in $C^{\ell-1}$; $u_{\ell+1}', v_{\ell+1}'$ the neighbours of $x_{\ell+1}'$ in $C^{\ell+1}$; and u_ℓ, v_ℓ the neighbours of x_ℓ in C^ℓ . We have that $V(C^{\ell-1} \cap C^{\ell+1}) = \{y_{\ell-1}'\}$, otherwise there would be a third short cycle in Γ containing x_ℓ . In particular, the vertices in $\{u_{\ell-1}', v_{\ell-1}', u_{\ell+1}', v_{\ell+1}', x_\ell'\}$ are pairwise distinct and $d \geq 5$. See Fig. 8 (a) and (b) for two drawings of this situation.

Let t_1, t_2, \dots, t_{d-4} denote the vertices in $N(x_\ell) - \{y_{\ell-1}, x_{\ell+1}, u_\ell, v_\ell\}$; see Fig. 8 (c). Consider a path $Q^i = t_i - y_{\ell-1}'$. Recall that Q^i has length at most $D - 1$. Since x_ℓ cannot be contained in a further short cycle, Q^i must be a $(D - 1)$ -path and go through a neighbour of $y_{\ell-1}'$ not contained in $\{u_{\ell-1}', v_{\ell-1}', u_{\ell+1}', v_{\ell+1}', x_\ell'\}$. Therefore, we have that $d \geq 6$ and, by the pigeonhole principle, that there are two paths Q^r and Q^s containing a common neighbour of $y_{\ell-1}'$. In this way, x_ℓ would be contained in a third short cycle, a

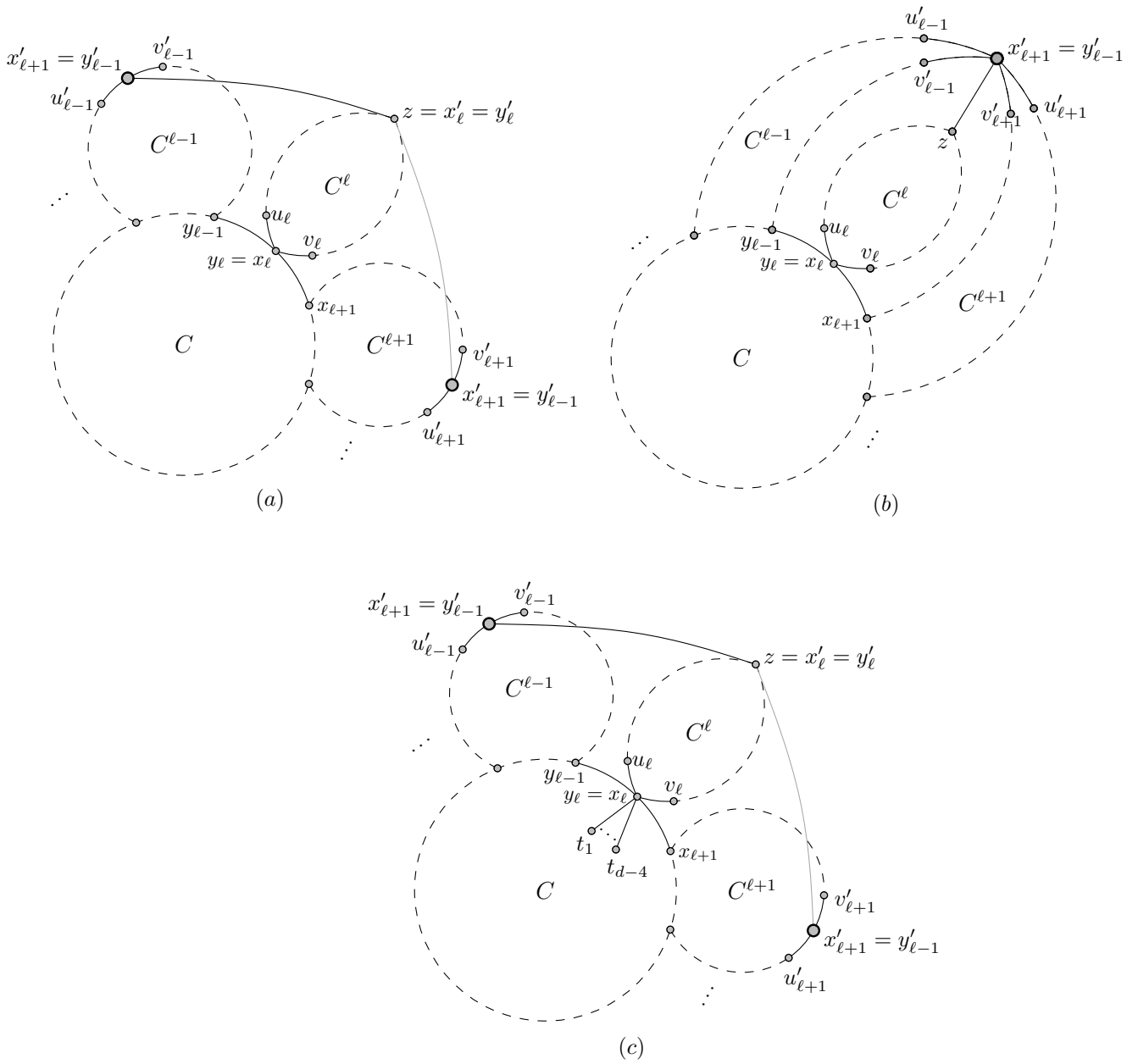


Figure 8: Auxiliary figure for Claim 1 of Lemma 4.2.

contradiction.

As a result, we conclude that the repeat graph Υ of C is indeed a $(2D - 2)$ -cycle C' as claimed. This completes the proof of Claim 1, and thus, of the lemma. \square

While not of primary interest, it is not difficult to prove now that the cycles C^1, C^2, \dots, C^k in the previous lemma are pairwise disjoint.

We call the aforementioned cycle C' the *repeat* of the cycle C in Γ , and denote it by $\text{rep}(C)$. Next some simple consequences of the Repeat Cycle Lemma follow.

Corollary 4.2 (Repeat Cycle Uniqueness) *If a short cycle C has a repeat cycle C' then C' is unique.*

Corollary 4.3 (Repeat Cycle Symmetry) *If $C' = \text{rep}(C)$ then $C = \text{rep}(C')$.*

Corollary 4.4 *Let C and C^1 be two short cycles in a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$ which intersect at a path I of length smaller than $D - 2$, and let $I' = \text{rep}^{C^1}(I)$. Then, the repeat cycle of C intersects C^1 at I' .*

Corollary 4.5 (Handy Corollary) *Let \mathcal{C} be a short cycle in a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$, and x, x' repeat vertices in \mathcal{C} . Let \mathcal{C}^1 and \mathcal{C}^2 be the other short cycles containing x and x' , respectively. Suppose that $I = \mathcal{C}^1 \cap \mathcal{C}$ is a path of length smaller than $D - 2$. Then, setting $y = \text{rep}^{\mathcal{C}^1}(x)$ and $y' = \text{rep}^{\mathcal{C}^2}(x')$, we have that y and y' are repeat vertices in the repeat cycle of \mathcal{C} .*

Proof. We denote the k neighbour cycles of \mathcal{C} as E^1, E^2, \dots, E^k and their respective intersection paths with \mathcal{C} as $I_1 = x_1 - y_1, I_2 = x_2 - y_2, \dots, I_k = x_k - y_k$ in such way that $\mathcal{C} = x_1 I_1 y_1 x_2 I_2 y_2 \dots x_k I_k y_k x_1$. For $1 \leq j \leq k$, we also denote $I'_j = x'_j - y'_j$, where $x'_j = \text{rep}^{E^j}(x_j)$ and $y'_j = \text{rep}^{E^j}(y_j)$.

Obviously, for some r, s ($1 \leq r, s \leq k$) we have that $\mathcal{C}^1 = E^r$, $\mathcal{C}^2 = E^s$, $x \in I_r$, $x' \in I_s$, $y \in I'_r$, and $y' \in I'_s$. We may assume $r < s$. By the Repeat Cycle Lemma, the vertices y and y' belong to the repeat cycle \mathcal{C}' of \mathcal{C} . Then the paths $x I_r y_r x_{r+1} I_{r+1} y_{r+1} \dots x_{s-1} I_{s-1} y_{s-1} x_s I_s x' \subset \mathcal{C}$ and $y I'_r y'_r x'_{r+1} I'_{r+1} y'_{r+1} \dots x'_{s-1} I'_{s-1} y'_{s-1} x'_s I'_s y' \subset \mathcal{C}'$ are both $(D - 1)$ -paths in Γ , and the corollary follows. \square

Proposition 4.2 *The set $S(\Gamma)$ of short cycles in a bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$ can be partitioned into sets $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$ and $S_{D-3}(\Gamma)$, where*

$S_{D-1}(\Gamma)$ is the set of short cycles whose intersections with neighbour cycles are $(D - 1)$ -paths;

$S_{D-2}(\Gamma)$ is the set of short cycles whose intersections with neighbour cycles are $(D - 2)$ -paths; and

$S_{D-3}(\Gamma)$ is the set of short cycles whose intersections with neighbour cycles are paths of length at most $D - 3$.

Proof. If Γ is one of the non-regular graphs in Fig. 2 the result trivially follows. We then assume that Γ is regular.

Let C be a short cycle in Γ . If C is contained in a Θ_{D-1} then, according to Proposition 4.1, all the intersections of C with its neighbour cycles are $(D - 1)$ -paths, in which case $C \in S_{D-1}(\Gamma)$.

Now suppose that, for some short cycle C^1 , $P_1 = C \cap C^1$ is a path of length $D - 2$. Note that all vertices in P_1 are saturated. Let v be an arbitrary vertex in P_1 , $v' = \text{rep}^C(v)$, and C^2 the short cycle other than C containing v' . Suppose that $P_2 = C \cap C^2$ is not a $(D - 2)$ -path. Then clearly P_2 cannot be a $(D - 1)$ -path, so it has length at most $D - 3$. But according to Corollary 4.4, the cycle $\text{rep}(C^2)$ intersects

C at exactly $\text{rep}^C(P_2)$, a proper subpath of P_1 . This implies that $\text{rep}(C^2)$ is a third short cycle containing the vertex v , a contradiction. Consequently, the intersections of C with its (exactly two) neighbour cycles are $(D - 2)$ -paths, and $C \in S_{D-2}(\Gamma)$.

Finally, if there is a short cycle intersecting C at a path of length at most $D - 3$ then, by the above reasoning, the intersections of C with all of its neighbour cycles are paths of length at most $D - 3$, and $C \in S_{D-3}(\Gamma)$. \square

The preceding result could be stated alternatively in term of vertices as follows:

Proposition 4.3 *The set $V(\Gamma)$ of vertices in a regular bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 3$ can be partitioned into sets $V_{D-1}(\Gamma)$, $V_{D-2}(\Gamma)$ and $V_{D-3}(\Gamma)$, where*

$V_{D-1}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-1}(\Gamma)$;

$V_{D-2}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-2}(\Gamma)$;

$V_{D-3}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-3}(\Gamma)$;

and $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$, $S_{D-3}(\Gamma)$ are defined as in Proposition 4.2. \square

5 Main results

5.1 Non-existence of subgraphs isomorphic to Θ_{D-1}

Theorem 5.1 *A bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 5$ does not contain a subgraph isomorphic to Θ_{D-1} .*

Proof. Suppose that Γ has a subgraph Θ isomorphic to Θ_{D-1} , with branch vertices a and b . Let p_1, p_2, p_3, p_4 and p_5 be as in Fig. 9 (a), and let q_1 be one of the neighbours of p_1 not contained in Θ .

Since all vertices of Θ are saturated, there cannot be a short cycle in Γ containing any of the incident edges of p_1, p_2, p_3, p_4 or p_5 which are not contained in Θ . According to this and by applying the Saturating Lemma, there is an additional short cycle D^1 in Γ such that q_1 and one of the neighbours of p_2 not contained in Θ (say q_2) are repeats in D^1 , and $D^1 \cap \Theta = \emptyset$. Analogously, in Γ there is an additional short cycle D^2 such that q_2 and one of the neighbours of p_3 not contained in Θ (say q_3) are repeats in D^2 , and $D^2 \cap \Theta = \emptyset$; an additional short cycle D^3 such that q_3 and one of the neighbours of p_4 not contained in Θ (say q_4) are repeats in D^3 , and $D^3 \cap \Theta = \emptyset$; and an additional short cycle D^4 such that q_4 and one of the neighbours of p_5 not contained in Θ (say q_5) are repeats in D^4 , and $D^4 \cap \Theta = \emptyset$. See Fig. 9 (b).

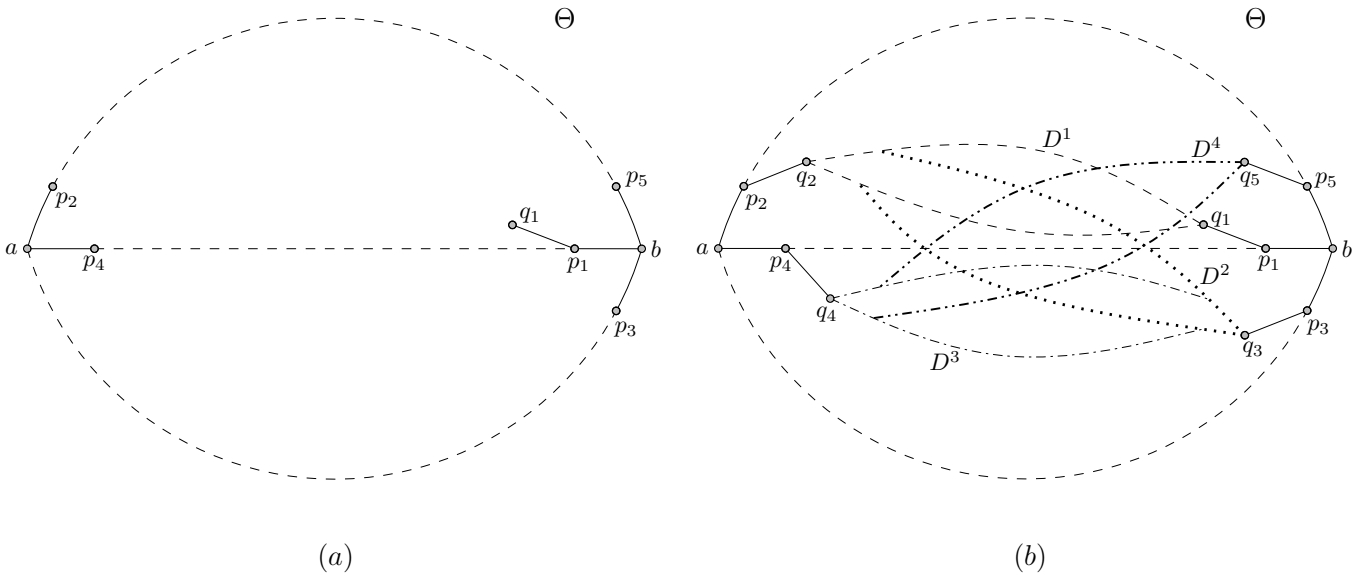


Figure 9: Auxiliary figure for Theorem 5.1

Note that $D^1 \cap D^2$ is a path of length at most $2 < D - 2$; otherwise for some vertex $t \in D^1 \cap D^2$ the closed walk $tD^1q_1p_1bp_3q_3D^2t$ would contain a cycle of length at most $2D - 2$ to which the vertex b would belong, a contradiction. For similar reasons, the intersection paths $D^2 \cap D^3$ and $D^3 \cap D^4$ all have length at most 2, with $2 \leq D - 2$. We now apply the Handy Corollary. By mapping the cycle D^2 to \mathcal{C} , the vertex q_2 to x , the vertex q_3 to x' , the cycle D^1 to \mathcal{C}^1 , the cycle D^3 to \mathcal{C}^2 , the vertex q_1 to y and the vertex q_4 to y' , we obtain that q_1 and q_4 are repeat vertices in the repeat cycle of D^2 . Therefore, since $q_4 \in D^4$, it follows that D^2 and D^4 are repeat cycles and $q_1 = q_5$. In this way, there would be a cycle $q_1p_1bp_5q_5$ in Γ of length $4 < 2D - 2$ (since $D \geq 5$), a contradiction to the fact that $g(\Gamma) = 2D - 2$. \square

Proposition 5.1 *The number N_{2D-2} of short cycles in a bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 5$ is given by the expression $\frac{2 \times (1 + (d-1) + \dots + (d-1)^{D-1}) - 4}{D-1}$.*

Proof. By Theorem 5.1, Γ does not contain a subgraph isomorphic to Θ_{D-1} . Then, according to Proposition 4.1, every vertex of Γ is contained in exactly two short cycles. We then count the number N_{2D-2} of short cycles of Γ . Since the order of Γ is $2 \times (1 + (d-1) + \dots + (d-1)^{D-1}) - 4$, we have that

$$N_{2D-2} = \frac{2 \times \left(2 \times (1 + (d-1) + \dots + (d-1)^{D-1}) - 4 \right)}{2D-2} = \frac{2 \times (1 + (d-1) + \dots + (d-1)^{D-1}) - 4}{D-1},$$

and the proposition follows. \square

5.2 Non-existence results on bipartite $(d, D, -4)$ -graphs

Since the number of short cycles in a graph Γ must be an integer, the expression obtained for N_{2D-2} in Proposition 5.1 already suffices to prove the non-existence of bipartite $(d, D, -4)$ -graphs for infinitely many pairs (d, D) .

Consider first the case in which $D - 1 = p^q$ is an odd prime power. Let $G = \{1, 2, \dots, p - 1\}$ be the multiplicative group of the field $\mathbb{Z}/p\mathbb{Z}$, let $d - 1 \not\equiv 0, 1 \pmod{p}$, and let H be the cyclic subgroup of G generated by $d - 1$. We observe that the sum of the elements of H is null \pmod{p} . Furthermore, since the order of H divides the order of G , it must also divide $p^q - 1 = D - 2$. Thus, we have

$$2 \times (1 + (d - 1) + \dots + (d - 1)^{D-1}) - 4 \equiv \begin{cases} -2 \pmod{p} & \text{if } d - 1 \equiv 0, 1 \pmod{p}, \\ 2(d - 1) - 2 \pmod{p} & \text{if } d - 1 \not\equiv 0, 1 \pmod{p}. \end{cases}$$

Therefore, it immediately follows

Corollary 5.1 *There is no bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 5$ such that $D - 1$ is an odd prime power.* \square

More generally, if p is an odd prime factor of $D - 1$ and $D - 1 \equiv r \pmod{p - 1}$, then

$$2 \times (1 + (d - 1) + \dots + (d - 1)^{D-1}) - 4 \equiv \begin{cases} -2 \pmod{p} & \text{if } d - 1 \equiv 0, 1 \pmod{p}, \\ 2 \frac{(d-1)^{r+1} - 1}{d-2} - 4 \pmod{p} & \text{if } d - 1 \not\equiv 0, 1 \pmod{p}; \end{cases}$$

Corollary 5.2 *There is no bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 6$ such that $d - 1 \equiv 0, 1 \pmod{p}$, where p is an odd prime factor of $D - 1$.* \square

It is also possible to examine completely the case of some small odd prime factors of $D - 1$. For example, it is not difficult to verify that, if $D - 1 = 3k$ then 3 does not divide $2 \times (1 + (d - 1) + \dots + (d - 1)^{D-1}) - 4$; thus,

Corollary 5.3 *There is no bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 5$ such that $D - 1 \equiv 0 \pmod{3}$.*

Now we turn to structural arguments to obtain other non-existence results.

Lemma 5.1 *Any two non-disjoint short cycles in a bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 7$ intersect at a path of length smaller than $D - 2$.*

Proof. Since Γ does not contain a graph isomorphic to Θ_{D-1} , it is only necessary to prove here that any two non-disjoint short cycles in Γ cannot intersect at a path of length $D - 2$.

Suppose, by way of a contradiction, that there are two short cycles C^1 and C^2 in Γ intersecting at a path I_1 of length $D - 2$. According to Proposition 4.2, C^2 is intersected by exactly two short cycles, namely C^1 and C^3 , at two independent $(D - 2)$ -paths. By repeatedly applying this reasoning and considering Γ is finite, we obtain a maximal length sequence $C^1, C^2, C^3, \dots, C^m$ of pairwise distinct short cycles in Γ such that C^i intersects C^{i+1} at a path I_i of length $D - 2$ ($1 \leq i \leq m - 1$), and $C^i \cap C^j = \emptyset$ for any $i, j \in \{1, \dots, m\}$ such that $2 \leq |i - j| \leq m - 2$.

Let us denote the paths $I_1 = x_1 - y_1, \dots, I_{m-1} = x_{m-1} - y_{m-1}$ in such way that, for $1 \leq i \leq m - 2$, $x_i \sim x_{i+1}$ and $y_i \sim y_{i+1}$ are edges in Γ . Also, let $x_0 \in N(x_1) \cap (C^1 - I_1)$, $y_0 \in N(y_1) \cap (C^1 - I_1)$, $x_m \in N(x_{m-1}) \cap (C^m - I_{m-1})$, and $y_m \in N(y_{m-1}) \cap (C^m - I_{m-1})$; see Fig. 10 (a). Set $I_0 = x_0 - y_0$ and $I_m = x_m - y_m$. Since the sequence $C^1, C^2, C^3, \dots, C^m$ is maximal and all the vertices in I_1, \dots, I_{m-1} are saturated, it follows that $I_0 = I_m$, and we have either $x_0 = x_m$ and $y_0 = y_m$ (Fig. 10 (b)), or $x_0 = y_m$ and $y_0 = x_m$ (Fig. 10 (c)).

If $x_0 = x_m$ and $y_0 = y_m$, then $m \geq 2D$; otherwise the cycle $x_1 x_2 \dots x_m x_1$ would have length at most $2D - 2$, contradicting the saturation of x_1 . If conversely $x_0 = y_m$ and $y_0 = x_m$ then $m \geq D$; otherwise the cycle $x_1 x_2 \dots x_m y_1 y_2 \dots y_m x_1$ containing x_1 would have length at most $2D - 2$, a contradiction as well. For our purposes, it is enough to state $m \geq D \geq 7$ in any case.

Let p_1 be the neighbour of y_1 on I_1 , and $p_{i+1} = \text{rep}^{C^{i+1}}(p^i)$ for $1 \leq i \leq 4$. Also, let q_1 be a neighbour of p_1 not contained in I_1 ; see Fig. 11 (a).

Since all vertices on I_1 are saturated, the edge $q_1 \sim p_1$ cannot be contained in a further short cycle. We apply the Saturating Lemma (by mapping C^2 to \mathcal{C} , p_1 to α , p_2 to α' , and q_1 to γ), and obtain in Γ an additional short cycle D^1 such that q_1 and one of the neighbours of p_2 not contained in I_2 (say q_2) are repeats in D^1 , and $D^1 \cap C^2 = \emptyset$. Analogously, for $2 \leq i \leq 4$ we obtain an additional short cycle D^i in Γ such that q_i and a neighbour of p_{i+1} not contained in I_{i+1} (say q_{i+1}) are repeats in D^i , and $D^i \cap C^{i+1} = \emptyset$; see Fig. 11 (b).

For $i = 1$ or 3 , $D^i \cap D^{i+1}$ cannot be a $(D - 2)$ -path; otherwise for some vertex $t_i \in D^i \cap D^{i+1}$, there would be a cycle $q_i p_i y_i y_{i+1} y_{i+2} p_{i+2} q_{i+2} D_{i+1} t_i D_i q_i$ of length at most $6 + D - 4 + D - 4$ (since $D - 2 \geq 5$), a contradiction to the fact that p_i is saturated and $g(\Gamma) = 2D - 2$. Analogously, $D^2 \cap D^3$ cannot be a $(D - 2)$ -path.

We now apply the Handy Corollary. By mapping the cycles D^2 to \mathcal{C} , D^1 to \mathcal{C}^1 and D^3 to \mathcal{C}^2 , and the vertices q_2 to x , q_3 to x' , q_1 to y , and q_4 to y' , it follows that the vertices q_1 and q_4 are repeat vertices in the repeat cycle of D^2 . Since $q_4 \in D^4$, D^2 and D^4 are repeat cycles and $q_5 = q_1$. In this way, we obtain a cycle $q_1 p_1 y_1 y_2 y_3 y_4 y_5 p_5 q_5$ in Γ of length $8 < 2D - 2$, a contradiction.

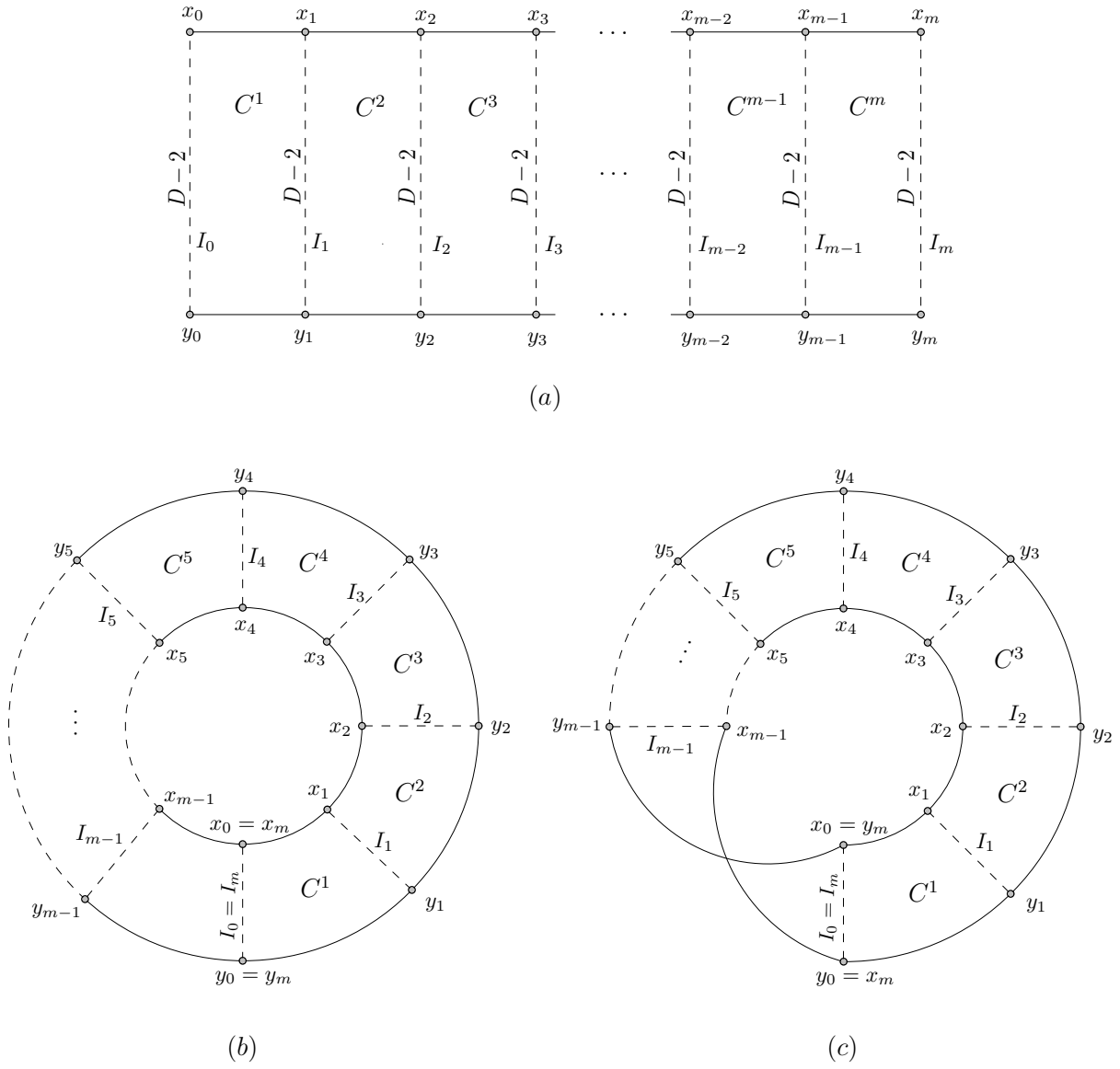


Figure 10: Auxiliary figure for Lemma 5.1

This completes the proof of the lemma. \square

Theorem 5.2 *There are no bipartite $(d, D, -4)$ -graphs for $d \geq 3$ and odd $D \geq 5$.*

Proof. The case $D = 5$ can be easily discarded by using Proposition 5.1, so we assume $D \geq 7$.

Suppose there is a bipartite $(d, D, -4)$ -graph Γ with $d \geq 3$ and odd $D \geq 7$. According to Lemma 5.1, any two non-disjoint short cycles in Γ intersect at a path of length smaller than $D - 2$, which means that every short cycle C in Γ has a repeat cycle C' (by the Repeat Cycle Lemma). Because of the uniqueness and symmetry of repeat cycles, the number N_{2D-2} of short cycles in Γ must be even.

However, since D is odd, the number $N_{2D-2} = \frac{2 \times (1 + (d-1) + \dots + (d-1)^{D-1}) - 4}{D-1}$ of short cycles in Γ is odd, a contradiction. \square

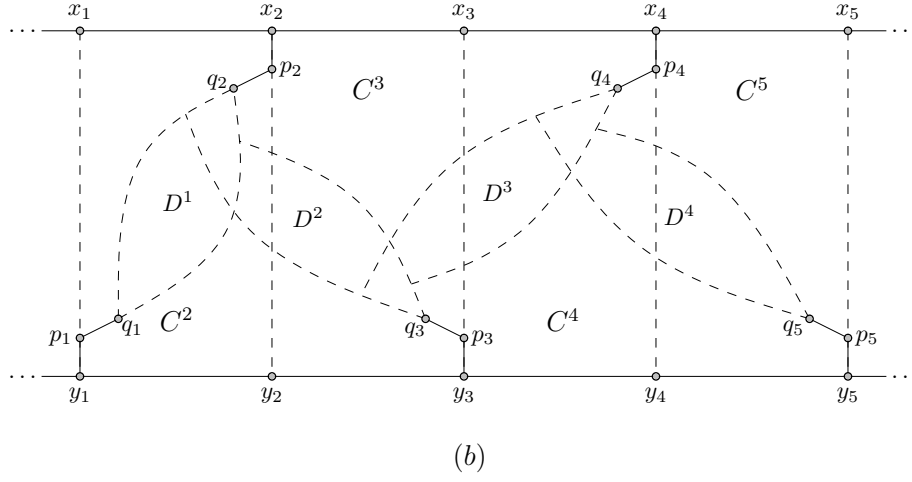
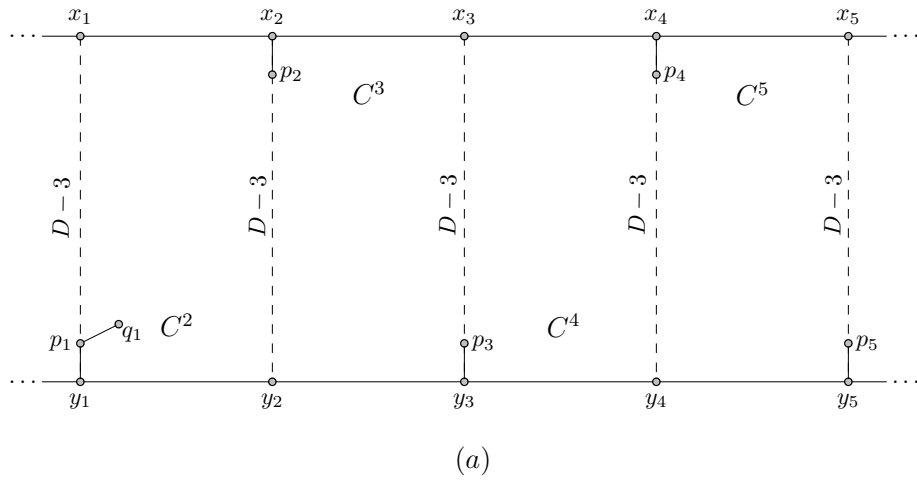


Figure 11: Auxiliary figure for Lemma 5.1

Furthermore, using Theorem 5.2 and Proposition 5.1 we have the following result.

Theorem 5.3 *There is no bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $5 \leq D \leq 187$.*

5.3 Non-existence of bipartite $(3, D, -4)$ -graphs with $D \geq 5$

In this section we complete the catalogue of bipartite $(3, D, -4)$ -graphs. Specifically, we prove the non-existence of bipartite $(3, D, -4)$ -graphs with even $D \geq 6$.

Lemma 5.2 *Any two non-disjoint short cycles in a bipartite $(3, D, -4)$ -graph Γ with $d \geq 3$ and $D \geq 7$ intersect at a path of length smaller than $D - 3$.*

Proof. By Lemma 5.1, it is only necessary to prove here that any two short cycles C and C^1 in Γ cannot intersect at a path $I = x - y$ of length $D - 3$. We proceed by contradiction. Let x' and y' be the

repeat vertices of x and y in C^1 , respectively. By Corollary 4.4, the repeat cycle C'' of C intersects C^1 at $I' = x' - y'$ (the repeat path of I in C^1); see Fig. 12. If we denote by z the neighbour of x on $C^1 - C$, then we have that the other short cycle containing z would also contain at least one of the vertices in $\{x, y'\}$, which contradicts the fact that x and y' are both saturated. \square

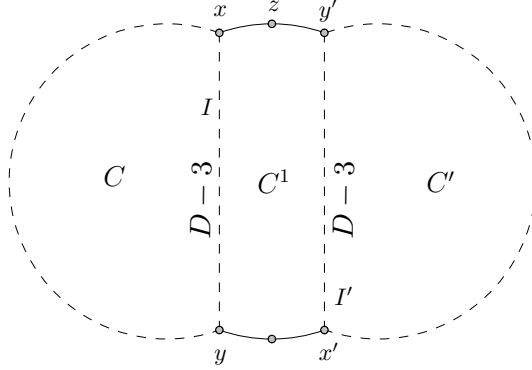


Figure 12: Auxiliary figure for Lemma 5.2

Theorem 5.4 *There are no bipartite $(3, D, -4)$ -graphs with even $D \geq 6$.*

Proof. Recall that Theorem 5.3 covers the case $D = 6$.

Let Γ be a bipartite $(3, D, -4)$ -graph with even $D \geq 8$, C^0 a short cycle in Γ , and x_0, x'_0 two repeat vertices in C^0 . Let x_1 and x'_1 be the neighbours of x_0 and x'_0 , respectively, not contained in C^0 . According to the Saturating Lemma, there is an additional short cycle C^1 containing x_1 and x'_1 such that $C^0 \cap C^1 = \emptyset$. Let y_1 be one of the neighbours of x_1 contained in C^1 , and $y'_1 = \text{rep}^{C^1}(y_1)$. Denote by x_2 and x'_2 the neighbours of y_1 and y'_1 , respectively, not contained in C^1 . Again by the Saturating Lemma, there is an additional short cycle C^2 such that $x'_2 = \text{rep}^{C^2}(x_2)$ and $C^1 \cap C^2 = \emptyset$. Since $d = 3$, we may assume that the other short cycle C containing x_0 also contains x_1 and a neighbour of x_0 in C_0 (say y_0). We first prove that $C^0 \cap C = y_0x_0$.

Claim 1. $C^0 \cap C = y_0x_0$.

Proof of Claim 1. Let y_0, z_0, y_2, y'_2 and z'_2 be as in Fig. 13 (a).

Consider a path $P = x'_2 - y_0$. If $y'_1 \in P$, then P would go through a neighbour of y'_1 contained in C^1 and there would be a cycle in Γ of length at most $2D - 4$. Therefore, we may assume $y'_2 \in P$. If $\{x'_2, y'_2, z'_2\} \subset V(P \cap C^2)$ then there would be a short cycle intersecting the cycle C^2 at a path of length $D - 3$, a contradiction to Lemma 5.2. Similarly, we have that $z_0 \notin P$. Also, P must be a $(D - 1)$ -path and $x_0 \notin P$; otherwise there would be a short cycle intersecting the cycle C^2 at a path of length $D - 2$, a contradiction to Lemma 5.1.

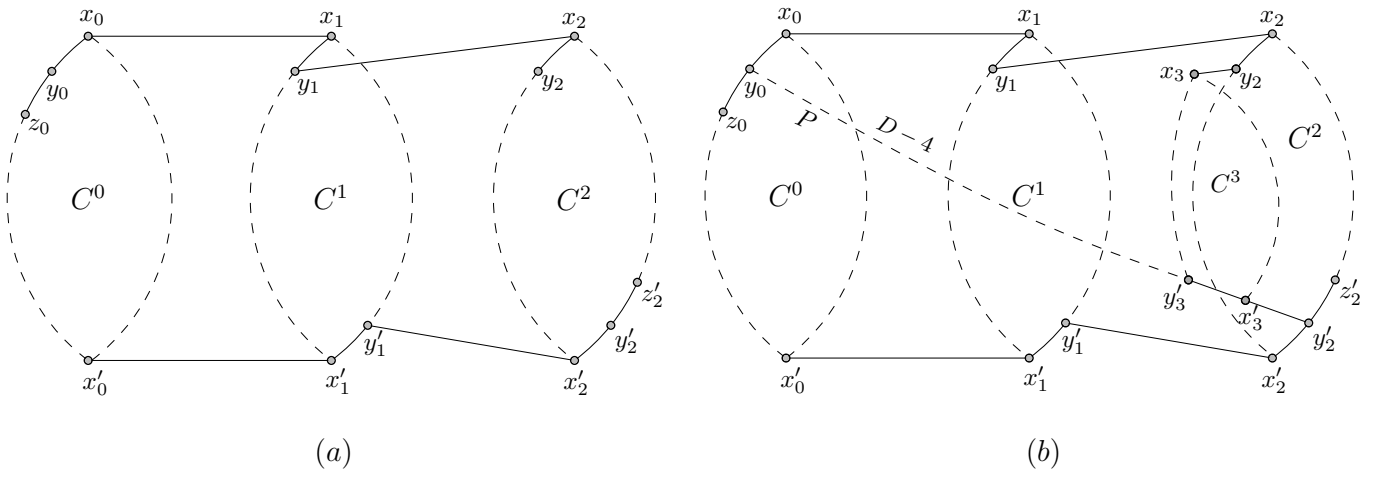


Figure 13: Auxiliary figure for Theorem 5.4.

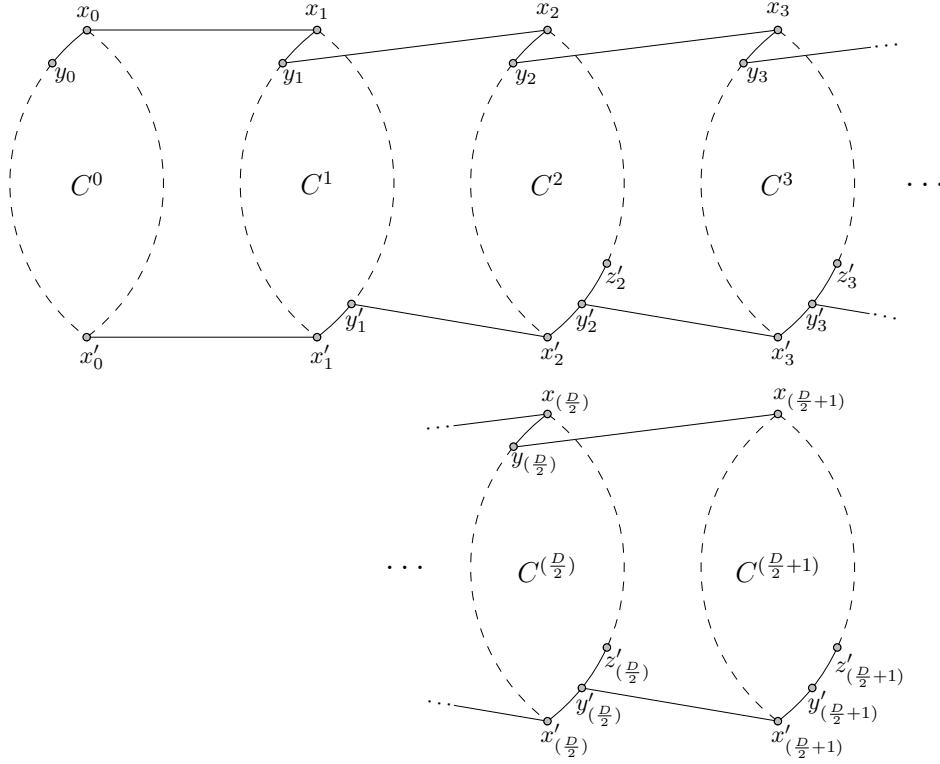


Figure 14: Auxiliary figure showing the sequence $C^0, C^1, \dots, C^{D/2+1}$ of short cycles.

For $3 \leq i \leq D/2 + 1$, let x_i and x'_i be the neighbours of y_{i-1} and y'_{i-1} , respectively, not contained in C^{i-1} , and let C^i be (in virtue of the Saturating Lemma) the additional short cycle disjoint from C^{i-1} which contains x_i and x'_i . Since P must go through x'_i , we denote by y'_i the neighbour of x'_i on $P \cap C^i$ and set $y_i = \text{rep}^{C^i}(y'_i)$. We now show that, if $i \neq D/2 + 1$, $P \cap C^i = x'_i y'_i$. Assume the contrary; that is, $P \cap C^i = x'_i y'_i z'_i$ (since $g(\Gamma) = 2D - 2$, $|V(P \cap C^i)| \leq 3$). In such case, there would be a short cycle $y_0 P z'_i C^i x_i y_{i-1} x_{i-1} y_{i-2} x_{i-2} \dots y_1 x_1 x_0 y_0$ intersecting C^i at a path of length $D - 3$, contradicting Lemma 5.2 (see Figures 13 (b) and 14). Consequently, $P \cap C^i = x'_i y'_i$ and P must go through a neighbour of y'_i not

contained in C^i .

In this way, for $3 \leq i \leq D/2 + 1$ we have $d(y_0, y'_i) = d(y_0, y'_{i-1}) - 2 = D - 2(i - 1)$, which means that $d(y_0, y'_{D/2+1}) = 0$. Since the cycle $C^{D/2+1}$ contains the vertices $y_0 \in C^0 \cap C \cap P$ and $x'_{D/2+1} \in P - C^0$, we have $C^{D/2+1} = C$, which implies that $C^0 \cap C = y_0 x_0$. \square

As the selection of C^0 and C was arbitrary, basically as a corollary of Claim 1 we have:

Claim 2. Any two non-disjoint short cycles in Γ intersect at an edge.

Finally, suppose that C^0 and C intersect at $y_0 x_0$, as stated by Claim 1. Let y'_0 be the repeat vertex of y_0 in C^0 ; then, by Corollary 4.4, the repeat cycle C' of C intersects C^0 at $y'_0 x'_0$ (the repeat path of $y_0 x_0$ in C^0). Setting $Q = x_0 C^0 y'_0 = x_0 w_1 \dots w_{D-3} y'_0$, we have Q is a path of length $D - 2$ with saturated endvertices (see Fig. 15). Therefore, by Claim 2, there exists a sequence $F^1, \dots, F^{D/2-2}$ of short cycles such that $F^i \cap C^0 = w_{2i-1} w_{2i}$. However, since D is even, the other short cycle containing w_{D-3} would also contain one of the vertices in $\{w_{D-4}, y'_0\}$, which contradicts the fact that w_{D-4} and y'_0 are both saturated. This completes the proof of Theorem 5.4. \square

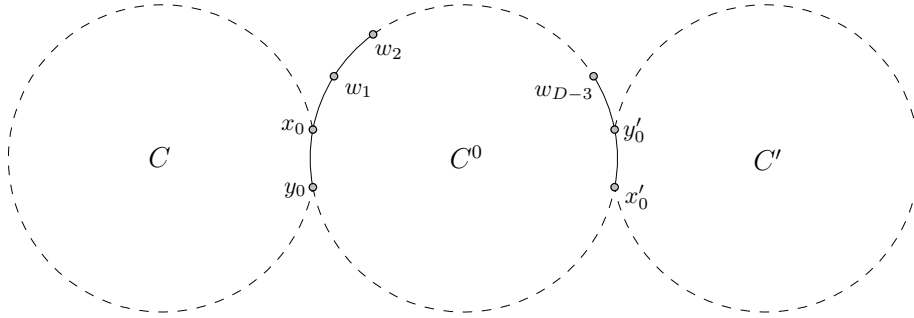


Figure 15: Auxiliary figure for Theorem 5.4.

Combining Theorems 5.2 and 5.4, we have that the only bipartite $(3, D, -4)$ -graphs with $D \geq 2$ are those depicted in Figures 2 and 3, completing in this way the catalogue of such graphs.

6 Conclusions

The main results obtained in this paper are summarised below.

First we stated important structural properties of bipartite $(d, D, -4)$ -graphs with $d \geq 3$ and $D \geq 3$. We found necessary conditions for the existence of bipartite $(d, D, -4)$ -graphs with $d \geq 3$ and $D \geq 5$, which allowed us to prove the non-existence of such graphs for infinitely many pairs (d, D) ; this included the case in which $D - 1$ is an odd prime power, and the case in which $D - 1 \equiv 0 \pmod{3}$. Afterwards, we went on to proving that bipartite $(d, D, -4)$ -graphs for $d \geq 3$ and odd $D \geq 5$ do not exist.

We completed the catalogue of bipartite $(\Delta, D, -4)$ -graphs with maximum degree $\Delta \geq 2$ and diameter $5 \leq D \leq 187$, which in turn completed the catalogue of bipartite $(\Delta, D, -\epsilon)$ -graphs with $\Delta \geq 2$, $5 \leq D \leq 187$, $D \neq 6$ and $0 \leq \epsilon \leq 4$.

Catalogue of bipartite (d, D, ϵ) -graphs with $\Delta \geq 2$, $5 \leq D \leq 187$ and $\epsilon = 0, 2$. In the case of $\epsilon = 0$, for $5 \leq D \leq 187$ and $d = 2$ the only Moore bipartite graphs are the $2D$ -cycles, whereas for $D = 6$ and $d \geq 3$ they are incidence graphs of generalised polygons. For other values of $d \geq 3$, $5 \leq D \leq 187$ and $\epsilon = 0$ there are no Moore bipartite graphs.

In the case of $\epsilon = 2$ the results of [14] combined with [5, 6] showed that there are no such graphs.

Catalogue of bipartite $(\Delta, D, -4)$ -graphs with $\Delta \geq 2$ and $5 \leq D \leq 187$. The path of length 5 is the only such graph.

Another important result of the paper is the completion of the catalogue of bipartite $(3, D, -\epsilon)$ -graphs with $D \geq 2$ and $0 \leq \epsilon \leq 4$.

Catalogue of bipartite $(3, D, 0)$ -graphs with $D \geq 2$. The cubic Moore bipartite graphs are the complete bipartite graph $K_{3,3}$ for $D = 2$, the unique incidence graph of the projective plane of order 2 for $D = 3$, the unique incidence graph of the generalised quadrangle of order 2 for $D = 4$, and the unique incidence graph of the generalised hexagon of order 2 for $D = 6$.

Catalogue of bipartite $(3, D, -2)$ -graphs with $D \geq 2$. There are only two non-isomorphic $(3, D, -2)$ -graphs with $D \geq 2$; a unique bipartite $(3, 2, -2)$ -graph (the claw graph), and a unique $(3, 3, -2)$ -graph, which is depicted in Fig. 1 (a).

Catalogue of bipartite $(3, D, -4)$ -graphs with $D \geq 2$. There exist no bipartite $(3, 2, -4)$ -graphs. When the diameter is 3, there are four non-isomorphic bipartite $(3, 3, -4)$ -graphs; all of them are shown in Figure 2. For diameter 4, there is a unique bipartite $(3, 4, -4)$ -graph, which is depicted in Fig. 3. The results of this paper, combined with those of [11], assert that there are no bipartite $(3, D, -4)$ -graphs with $D \geq 5$, outcome that gives an alternative proof of the optimality of the known bipartite $(3, 5, -6)$ -graph (see [4]).

6.1 Bipartite $(d, D, -4)$ -graphs with $d \geq 4$ and $D = 3, 4$

The main results in this paper did not include bipartite $(d, D, -4)$ -graphs with $d \geq 4$ and $D = 3, 4$. However, we believe that the structural properties of these graphs provided in Section 4 could bear more conclusive results on such diameters.

For instance, by using Proposition 4.1, Lemma 4.1 (Saturating Lemma), Lemma 4.2 (Repeat Cycle Lemma), Proposition 4.2, and Proposition 4.3, we were able to prove analytically the uniqueness of the two bipartite $(4, 3, -4)$ -graphs depicted in Fig. 4. We also think there should be no major difficulty to complete as well –in a very similar manner– the catalogue of bipartite $(5, 3, -4)$ -graphs, which has so far as a unique element, the graph in Fig. 5.

Unfortunately, the final ideas used in the paper cannot be easily extended to cover bipartite $(d, D, -4)$ -graphs with $d \geq 4$ and $D = 3, 4$. With our current approach we cannot have Theorem 5.1 for $D = 3, 4$. In Theorem 5.1 the intersection paths $D^1 \cap D^2$, $D^2 \cap D^3$ and $D^3 \cap D^4$ have length at most 2, and for us to apply the Repeat Cycle Lemma we need the lengths of such paths to be less than $D - 2$. Indeed, the graph in Fig. 4 (a) offers a good illustration of this. Even if we had Theorem 5.1, something similar would occur with Lemma 5.1; see the graphs in Fig. 4 (b) and Fig. 5.

6.2 Remarks on the upper bound for $N^b(d, D)$

Our results improve the upper bound on $N_{d,D}^b$ for many combinations of d and D . Recall that a bipartite $(d, D, -5)$ -graph Γ with $d \geq 3$ and $D \geq 5$ must be regular (by Proposition 3.1) and thus cannot exist. Indeed, the upper bound of $M^b(d, D) - 6$ for $N^b(d, D)$ has been established in the following cases:

- $d \geq 3$ and $D \geq 8$ such that $D - 1$ is an odd prime power.
- $d \geq 3$ and $D \geq 7$ such that $d - 1 \equiv 0, 1 \pmod{p}$, where p is an odd prime factor of $D - 1$.
- $d \geq 3$ and $D \geq 5$ such that $D \equiv 1, 3, 4, 5 \pmod{6}$.
- $d \geq 3$ and $5 \leq D \leq 187$ ($D \neq 6$).
- $d = 3$ and any $D \geq 5$ ($D \neq 6$).

Finally, we feel that the next conjectures are valid.

Conjecture 6.1 *There is no bipartite $(d, D, -4)$ -graph with $d \geq 3$ and $D \geq 5$.*

Conjecture 6.2 *For natural numbers $d \geq 3$ and $D \geq 5$ such that $D \neq 6$, $N^b(d, D) \leq M^b(d, D) - 6$.*

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